

JOURNAL OF ALGEBRA, 1, 367-373 (1964)

Projective Modules over Free Groups Are Free*

HYMAN BASS

*Department of Mathematics, Columbia University, New York, New York**Communicated by P. M. Cohn*

Received June 11, 1964

INTRODUCTION

The title refers to:

THEOREM 1. *Let π be a free group (or monoid), let R be a principal ideal domain, and let $A = R\pi$, the group ring. Then finitely generated, projective, right (or left) A -modules are free.*

Remarks: (1) Theorem 1 is related to a question raised by Eilenberg and Ganea [4].

(2) If π is free on one generator then A is commutative, and the theorem is due to Seshadri [7].

(3) If R is a field it is a theorem of P. M. Cohn [3].

Suppose $A = R[t]$ is a polynomial ring in one variable. Seshadri's argument in this case makes essential use of the fact that $(R/\mathfrak{G})[t]$ is a euclidean ring for all maximal ideals \mathfrak{G} in R . This, of course, limits his method to a single variable. However, Cohn's recent work (see, e.g., [1-3]) has revealed the remarkable fact that a polynomial ring over a field in *any* number of variables is a "euclidean ring" (see Section 1), *provided* the variables do not commute. In particular, Cohn's results supply the hypotheses, in the setting of Theorem 1, of a suitably axiomatized version (Theorem 2) of Seshadri's argument, and this is how Theorem 1 is proved.

The last section contains a result concerning projective modules, which need not be finitely generated, over semihereditary rings. This is the non-commutative extension of a theorem (and proof) of Kaplansky (see [5], Theorem 3). Its relevance here is that it would yield some extensions of

* This work has been partially supported by the National Science Foundation under Grant No. NSF GP 1904.

1. GENERALIZED EUCLIDEAN RINGS

Cohn's results, provided one could establish the (probable) fact that the global dimension of a free product is the supremum of those of the factors.

This section records some results of P. M. Cohn to be used in the sequel.

For a ring A , $GL(n, A)$ denotes the group of invertible $n \times n$ matrices over A and $E(n, A)$ the subgroup generated by all elementary matrices, i.e. those differing from the identity in a single, off diagonal, coordinate. We identify

$$\alpha \in GL(n, A) \quad \text{with} \quad \begin{pmatrix} \alpha & 0 \\ 0 & 1_m \end{pmatrix} \in GL(n+m, A).$$

A is a *generalized euclidean ring* if, given $a_1, \dots, a_n, b_1, \dots, b_n$ in A with some $b_i \neq 0$ and $\sum a_i b_i = 0$, then there is an $\epsilon \in E(n, A)$ such that $\alpha\epsilon$ has at least one coordinate zero, where $\alpha = (a_1, \dots, a_n)$.

The proof of the next Proposition requires only trivial modifications of the arguments proving Theorem 2.6 in [3].

PROPOSITION 1.1 (P. M. Cohn). *A generalized euclidean ring, A , has the following properties:*

- (a) A^0 (the opposite ring) is generalized euclidean.
- (b) A finitely generated submodule of a projective right (or left) A -module is free.
- (c) Any n generators of the free module, A^n , are a basis. In particular, two bases have the same cardinality.
- (d) $GL(n, A) = GL(1, A) \cdot E(n, A)$ for all $n \geq 1$. I.e. an invertible matrix can be reduced, by elementary column operations, to a matrix of the form

$$\begin{pmatrix} u & 1 & & 0 \\ & & \ddots & \\ & & & 1 \\ 0 & & & \end{pmatrix}, \quad u \in A.$$

A euclidean ring in the classical sense is easily seen to be generalized euclidean (c.f. [8], §108). Thus, \mathbf{Z} and $k[t]$, k a field, are (commutative) examples. Moreover, any valuation ring is generalized euclidean. Non commutative examples are supplied by the next theorem.

THEOREM 1.2 (P. M. Cohn). *Let π be a free group or monoid, and let k be a field. Then $k\pi$ is generalized euclidean.*

This theorem is deduced as follows: $k\pi$ above can be regarded as the "free product" over k of algebras of the type $k[t]$ and $k[t, t^{-1}]$. One therefore need only know that a free product, over a field, of generalized euclidean rings is

again such. With a slightly weaker condition than generalized euclidean, this is just Theorem 4.2 of [3]. However, Cohn has recently shown (unpublished) that his proof can be adapted to generalized euclidean rings as well.

2. ELEMENTARY AUTOMORPHISMS

Let $L = L_1 \oplus \cdots \oplus L_n$ be a right A -module decomposed into a direct sum of submodules, and let $D = (L_1, \cdots, L_n)$ designate this decomposition. Suppose $f: L \rightarrow L$ is an endomorphism such that, for some i and j , $i \neq j$, $f(L_i) \subset L_j$, and $f(L_k) = 0$ for $k \neq i$. Then $f^2 = 0$, so $e = 1_L + f$ is an automorphism. Such automorphisms will be called D -elementary and $E(D, L)$ denotes the group they generate.

For example, if $L = A^n$, and $A^n = A \oplus \cdots \oplus A$ is the standard decomposition, then $E(D, L)$ is what, in the last section, was denoted by $E(n, A)$ (after using the standard basis to obtain a matrix representation).

Now let \mathfrak{A} be a two sided ideal, and write $M' = M/M\mathfrak{A}$, for a right A -module, M .

PROPOSITION 2.1. *Given L and $D = (L_1, \cdots, L_n)$ as above, let*

$$D' = (L'_1, \cdots, L'_n)$$

be the induced decomposition of L' . Then if L is A -projective, the natural homomorphism

$$E(D, L) \rightarrow E(D', L')$$

is surjective.

Proof: It suffices to lift generators, $1_{L'} + f$, as above. Say $f(L'_i) \subset L'_j$, $i \neq j$, and $f(L'_k) = 0$ for $k \neq i$. Then it clearly suffices to lift the induced homomorphism $L'_i \rightarrow L'_j$, i.e. to find $h: L_i \rightarrow L_j$ rendering commutative the square

$$\begin{array}{ccc} L_i & \xrightarrow{h} & L_j \\ \downarrow & & \downarrow \\ L'_i & \xrightarrow{f} & L'_j \end{array}$$

where the verticals are canonical projections. The existence of h follows since L_i is projective.

3. THE MAIN THEOREM

In this section, " A -module" means "finitely generated right A -module."

THEOREM 2. *Let R be a Dedekind domain with field of quotients K , and let A be an R -algebra satisfying the following conditions:*

- (i) A is R -projective.

- (ii) If P is a projective A -module then $PK(=P \otimes_R K)$ is AK -free.
 (iii) If \mathfrak{G} is a maximal ideal of R then A/\mathfrak{G} is a generalized euclidean ring.

Then a projective A -module, P , is induced from R . I.e., $P \cong A \otimes_R P_0$, with P_0 a (projective) R -module.

COROLLARY. If R above is a principal ideal domain, then projective A -modules are free.

Remarks. (1) P_0 above is known (see, e.g., [5]) to be a direct sum of a free module, and a module of rank one, the latter being isomorphic to an ideal, $\mathfrak{A} \neq 0$, in R . Moreover, $M \otimes_R \mathfrak{A} \cong M\mathfrak{A}$, so that induced A -modules are stable under multiplication by ideals of R .

(2) Suppose $A = R\pi$, as in Theorem 1. Then (i) is satisfied. Let $k = K$, or R/\mathfrak{G} for a maximal ideal \mathfrak{G} . Then $k\pi$ is generalized euclidean (condition (iii)) by Theorem 1.2, and hence, by (1.1)(b), projective $k\pi$ -modules are free (condition (ii)). Thus, Theorem 1 is a corollary of Theorem 2.

Proof of Theorem 2: Since A is R -projective, P is likewise, so that $P \subset PK$. Since, by hypothesis, PK is AK -free, it has a (finite) basis. Clearing denominators, we may find such a basis in P . Let L be the (free) A -module generated by this basis. Since $LK \subset PK$, P/L is a torsion R -module. Let $\mathfrak{A} \neq 0$ in R annihilate a finite A -generating set of P/L . Then clearly $P\mathfrak{A} \subset L$. This shows the existence of $L \subset P$ and $\mathfrak{A} \neq 0$ in R such that

- (1) L is induced, and
 (2) $P\mathfrak{A} \subset L$.

Since R is noetherian we may choose, among all such pairs, one for which \mathfrak{A} cannot be enlarged. It remains to show then that $\mathfrak{A} = R$, and hence that $P = L$ is induced, as the theorem claims.

If $\mathfrak{A} \neq R$ write $\mathfrak{A} = \mathfrak{G}\mathfrak{L}$ with \mathfrak{G} maximal. For an A -module, M , write $M' = M/\mathfrak{G}M$. Since L is induced, $L = L_1 \oplus \cdots \oplus L_n$ with each L_i induced from a projective module of rank one (remark 1 above), and it follows that $L'_i \cong A'$ for each i . The inclusion $L \subset P$ induces $f: L' \rightarrow P'$. Since P' is projective over the generalized euclidean ring A' , (1.1) (b) shows that $f(L')$ is A' -free. Hence L' has the form $L' = P' \oplus G'$ with $G' = \ker f$. Again (1.1)(b) implies F' and G' are A' -free, so we can write $F' = F'_1 \oplus \cdots \oplus F'_r$, and $G' = G'_1 \oplus \cdots \oplus G'_s$, with each F'_i and $G'_j \cong A'$. Then (1.1)(c) implies $r + s = n$. Hence, there is an automorphism, δ' , of L' , such that $\delta'(L'_i) = F'_i$, $1 \leq i \leq r$, and $\delta'(L'_{r+i}) = G'_i$, $1 \leq i \leq s$. Let $L'_i = e_i A'$, and represent δ' by an element of $GL(n, A')$, relative to the basis, e_1, \dots, e_n . Then, by (1.1)(d), we can write $\delta' = \delta_0 \epsilon'$, where ϵ' is represented by an element of $E(n, A')$, and $\delta_0(e_1) = e_1 u$, $\delta_0(e_i) = e_i$ ($i > 1$). Replacing δ' by $\delta_0^{-1} \delta'$, we can

assume $\delta' = \epsilon'$. If $D' = (L'_1, \dots, L'_n)$ then, in the notation of Section 2, $\epsilon' \in E(D', L')$. Hence, ϵ' is the reduction, mod \mathfrak{G} , of an $\epsilon \in E(D, L)$, by virtue of Proposition 2.1.

Let $F = \epsilon(L_{r_1} \oplus \dots \oplus L_{r_s})$ and $G = \epsilon(L_{r_{s+1}} \oplus \dots \oplus L_{r_n})$. Then $L = F \oplus G$, and this reduces, mod \mathfrak{G} , to the decomposition $L' = F' \oplus G'$ above. The fact that $f(G') = 0$ and $f|F'$ is injective now translates into

$$G \subset P\mathfrak{G} \quad \text{and} \quad F \cap P\mathfrak{G} = F\mathfrak{G}. \quad (*)$$

Let $Q = F \oplus G\mathfrak{G}^{-1}$. Then $L \subset Q \subset P$, and Q is induced. The desired contradiction will be achieved when we show that $P\mathfrak{Q} \subset Q$, since $\mathfrak{G} \not\subset_{\mathfrak{H}} \mathfrak{Q}$, and we chose \mathfrak{A} maximal.

Recall that $\mathfrak{A} = \mathfrak{G}\mathfrak{Q}$, and $P\mathfrak{A} \subset L$. To show that $P\mathfrak{Q} \subset Q$ it suffices to check it locally, so we may assume $\mathfrak{G} = (p)$ is principal. Let $x \in P\mathfrak{Q}$. Then $xp \in P\mathfrak{A} \subset L = F \oplus G$, so $xp = y \oplus z$, $y \in F$, $z \in G$. $G \subset P\mathfrak{G} = Pp$, by (*), so $z = z_0p$, $z_0 \in P$. Then $y = (x - z_0)p \in Pp \cap F = Fp$, by (*) again; say $y = y_0p$, $y_0 \in F$. Then $xp = y_0p \oplus z_0p$, so $x = y_0 \oplus z_0 \in F \oplus Gp^{-1} = Q$, as claimed.

4. SEMIHEREDITARY RINGS

If P is a left (right) A -module, then $P^* = \text{Hom}_A(P, A)$ is a right (left) A -module. If $x \in P$, P a right A -module, then

$$o_P(x) = \{f(x) \mid f \in P^*\}$$

is evidently a left ideal in A .

PROPOSITION 4.1. *Let P be a projective right A -module and $x \in P$. Let $Q = \{y \in P \mid f \in P^*, f(x) = 0 \Rightarrow f(y) = 0\}$. Then:*

- (i) $o_P(x)$ is a finitely generated left ideal.
- (ii) $Q \cong o_P(x)^*$
- (iii) Q is a direct summand of $P \Leftrightarrow o_P(x)$ is projective. In this case Q is (by (ii)) finitely generated.

Proof: Say $P \oplus P' = F$ is free, and $x \in F_0$, F_0 generated by a finite subset of a basis for F . Then it is clear that $o_P(x) = o_{F_0}(x)$, and that $Q = \{y \in F_0 \mid f \in F_0^*, f(x) = 0 \Rightarrow f(y) = 0\}$. Moreover, Q is a direct summand of P and F_0 simultaneously. Hence we may assume P is finitely generated and free.

(i) $o_P(x)$ is evidently the left ideal generated by the coordinates of x relative to any basis of P .

(ii) Let $f: P \rightarrow M$ be the canonical projection onto $M = P/xA$. It is easily checked that $Q = \ker(f^{**}: P^{**} \rightarrow M^{**})$. Let $g: A \rightarrow P$ by $g(a) = xa$, so that $A \xrightarrow{g} P \xrightarrow{f} M \rightarrow 0$ is exact. Then $0 \rightarrow M^* \xrightarrow{A^*} P^* \xrightarrow{g^*} A^*$ is exact, and $\text{im } g^* = o_P(x)$, after identifying A^* with A in the usual fashion. We thus obtain an exact sequence

$$0 \rightarrow M^* \xrightarrow{f^*} P^* \xrightarrow{h} o_P(x) \rightarrow 0,$$

and this dualizes to the exact sequence,

$$0 \rightarrow o_P(x)^* \xrightarrow{h^*} P^{**} \xrightarrow{f^{**}} M^{**}.$$

Finally, then, $Q = \ker f^{**} = \text{im } h^* \cong o_P(x)^*$.

(iii) If $o_P(x)$ is projective then h has a right inverse, so h^* has a left inverse, i.e. $Q = \text{im } h^*$ is a direct summand.

Conversely, suppose h^* has a left inverse. Then $h^{**}: P^{***} \rightarrow o_P(x)^{**}$ has a right inverse, say k ; $h^{**}k = \text{id}$ on $o_P(x)^{**}$. Then the composite, $o_P(x) \rightarrow o_P(x)^{**} \xrightarrow{k} P^{***} = P^*$, in the commutative diagram

$$\begin{array}{ccc} P^* & \longrightarrow & o_P(x) \\ \downarrow = & & \downarrow \\ P^{***} & \xrightarrow{h^{**}} & o_P(x)^{**} \\ & \xleftarrow{k} & \end{array}$$

is the desired right inverse for h , provided that $o_P(x) \rightarrow o_P(x)^{**}$ is injective. The latter is the case because the inclusion, $i: o_P(x) \rightarrow A$, is an element of $o_P(x)^*$ killed by no non zero element of $o_P(x)$ in $o_P(x)^{**}$.

THEOREM 3. *Let A be a left semi-hereditary ring (i.e. finitely generated left ideals are projective). Then a projective right A -module, P , is a direct sum of finitely generated modules, each isomorphic to the dual of a finitely generated left ideal.*

Proof: If $x \in P$ then $o_P(x)$ is a finitely generated left ideal (4.1)(i), and hence projective. By (4.1)(ii) and (iii), $x \in Q$, with Q a direct summand of P isomorphic to $o_P(x)^*$.

Now by Kaplansky's argument (see [6], Lemma 1 and Theorem 3), the theorem follows.

COROLLARY. *If every finitely generated left ideal in a ring A is free, then every projective right A -module is free.*

REFERENCES

1. COHN, P. M., Non-commutative unique factorization domains. *Trans. Am. Math. Soc.* **109** (1963), 313-331.
2. COHN, P. M., Rings with a weak algorithm. *Trans. Am. Math. Soc.* **109** (1963), 332-356.
3. COHN, P. M., Free ideal rings. *J. Algebra* **1** (1964), 47-69.
4. EILENBERG, S., AND GANEA, T., On the Lusternik-Schnirelmann category of abstract groups. *Ann. Math.* **65** (1957), 517-518.
5. KAPLANSKY, I., Modules over Dedekind rings and valuation rings. *Trans. Am. Math. Soc.* **72** (1952), 327-340.
6. KAPLANSKY, I., Projective modules. *Ann. Math.* **68** (1958), 372-377.
7. SESUADRI, C. S., Triviality of vector bundles over the affine space K^2 . *Proc. Natl. Acad. Sci., U.S.* **44** (1958), 456-458.
8. VAN DER WAERDEN, B. L., "Modern Algebra," Vol. II. Ungar, New York, 1950.